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NOTE

ON A PROBLEM OF KATONA ON MINIMAL COMPLETELY SEPARATING SYSTEMS WITH RESTRICTIONS

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Let S be a set of n elements, and k a fixed positive integer $<\frac{1}{2}n$. Katona's problem is to determine the smallest integer m for which there exists a family $\mathscr{A}=\{A_1,\ldots,A_m\}$ of subsets of S with the following property: $|A_i|\leqslant k$ $(i=1,\ldots,m)$, and for any ordered pair x_i, $x_j\in S$ $(i\neq j)$ there is $A_l\in\mathscr{A}$ such that $x_i\in A_l$, $x_j\notin A_l$. It is given in this note that $m=\lceil 2n/k\rceil$ if $n>\frac{1}{2}k^2\geqslant 2$.

Given a family $\mathscr{A}=\{A_1,\ldots,A_m\}$ of subsets of $S=\{x_1,\ldots,x_n\}$, we define the dual of $\mathscr{A}$ as the family $\mathscr{A}^*=\{X_1,\ldots,X_n\}$ of subsets of $\{a_1,\ldots,a_m\}$, where $X_i=\{a_j\mid A_j\in\mathscr{A},\ x_i\in A_j\}$, $i=1,\ldots,n$.

A family $\mathscr{A}=\{A_1,\ldots,A_m\}$ of subsets of $S=\{x_1,\ldots,x_n\}$ is called a completely separating system if for any ordered pair x_i, $x_j\in S$ $(i\neq j)$ there exists $A_l\in\mathscr{A}$ such that $x_i\in A_l$, $x_j\notin A_l$.

It is easy to show

Claim. *A family $\mathscr{A}=\{A_1,\ldots,A_m\}$ on $S=\{x_1,\ldots,x_n\}$ is a completely separating system iff its dual $\mathscr{A}^*=\{X_1,\ldots,X_n\}$ is a Sperner family, i.e., no one member of $\mathscr{A}^*$ contains another.*

The minimal cardinality of a completely separating system on an n-element set was established by Spencer [3].

Let S be a set of n elements, and k a fixed positive interger $<\frac{1}{2}n$. In 1973, Katona [2] posed the problem of determining the smallest integer m for which there exists a completely separating system $\mathscr{A}=\{A_1,\ldots,A_m\}$ on S satisfying $|A_i|\leqslant k$, $i=1,\ldots,m$.

To avoid the trivial case $k=1$, we shall suppose that $k\geqslant 2$. The problem can be solved provided n is large enough (relatively to k).

Theorem 1. *Let m denote the minimal cardinality of such a system required by Katona's problem for $S=\{x_1,\ldots,x_n\}$. Then*

$$m=\lceil 2n/k\rceil \quad \textit{if } n>\tfrac{1}{2}k^2,$$

where $\lceil\gamma\rceil$ denotes the smallest integer $\geqslant\gamma$.

Proof. First let us show

$$m \geqslant \lceil 2n/k \rceil. \tag{1}$$

By the definition of m, there exists a completely separating system $\mathscr{A} = \{A_1, \ldots, A_m\}$ on S. Consider its dual $\mathscr{A}^* = \{X_1, \ldots, X_n\}$. Supposing $X_j = \{a_i\}$, i.e., a singleton, then $A_i = \{x_j\}$ by the Claim. Thus we may assume without loss of generality that (A_i and X_j being renumbered if necessary)

$$\begin{aligned} A_i &= \{x_i\}, \quad i = 1, \ldots, l, \\ X_j &= \{a_j\}, \quad j = 1, \ldots, l, \\ |X_j| &\geqslant 2, \quad j = l+1, \ldots, n. \end{aligned}$$

Counting the number of pairs (A_i, X_j), where $x_j \in A_i$, $i = l+1, \ldots, m$, $j = l+1, \ldots, n$, in two different ways, we have

$$\sum_{i=l+1}^{m} |A_i| = \sum_{j=l+1}^{n} |X_j|.$$

Since $|A_i| \leqslant k$ and $|X_j| \geqslant 2$, $j = l+1, \ldots, n$, hence

$$k(m-l) \geqslant 2(n-l).$$

As $k \geqslant 2$, we obtain $km \geqslant 2n$, yielding (1).

So to prove the theorem it suffices to show

$$m \leqslant \lceil 2n/k \rceil \quad \text{if } n > \tfrac{1}{2}k^2. \tag{2}$$

Let $p = \lceil 2n/k \rceil$, then $p \geqslant k+1$. It is easy to construct a simple graph $G = (V, E)$ with vertex-set $V(G)$ and edge-set $E(G)$ such that

$$|V(G) = p, \qquad |E(G)| = n, \quad \text{and} \quad \Delta(G) \leqslant k,$$

where $\Delta(G)$ denotes the maximum degree of vertices of G

Now for each vertex v_i of G, $i = 1, \ldots, p$, set

$$A_i = \{e_j \mid e_j \in E(G),\ e_j \text{ is incident with } v_i\}.$$

We leave it to the reader to show the family $\mathscr{A} = \{A_1, \ldots, A_p\}$ is a completely separating system for the n-element set $E(G)$. Therefore (2) holds. The proof is completed.

A slight generalization of Katona's problem can be made. Let $k_1, \ldots, k_m$ be a given sequence of integers with $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m \geqslant 2$. What is the largest integer n for which there exists a Sperner family $\mathscr{X} = \{X_1, \ldots, X_n\}$ on $\{a_1, \ldots, a_m\}$ such that a_i belongs to at most k_i members of $\mathscr{X}$ $(i = 1, \ldots, m)$?

Based on the Claim, the problem of Katona is equivalent to the special case of the generalization when $k_i = k$, $i = 1, \ldots, m$.

A similar argument can be used to prove

Theorem 2. *Given an integer sequence $k_1, k_2, \ldots, k_m$ with $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m \geqslant 2$, let $\mathcal{N}$ denote the maximal cardinality of such a Sperner family on an m-element set, required by the generalized problem. If the sequence $k_1, k_2, \ldots, k_m$ or $k_1 - 1, k_2, k_3, \ldots, k_m$ is graphic, then*

$$\mathcal{N} = \left\lfloor \sum_{i=1}^{m} \tfrac{1}{2} k_i \right\rfloor,$$

where $\lfloor \gamma \rfloor$ denotes the largest integer $\leqslant \gamma$.

Remark. A sequence $d_1, d_2, \ldots, d_m$ is graphic if there is a simple graph with degree sequence $d_1, d_2, \ldots, d_m$. Suppose that $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m$ and $\sum_{i=1}^{m} d_i$ is even, the theorem of Erdös and Gallai [1] says that $d_1, d_2, \ldots, d_m$ is graphic iff for each interger γ, $1 \leqslant \gamma \leqslant m$,

$$\sum_{i=1}^{\gamma} d_i \leqslant \gamma(\gamma - 1) + \sum_{i=\gamma+1}^{m} \min\{\gamma, d_i\}.$$

References

[1] P. Erdös and T. Gallai, Graphs with prescribed degrees of vertices (in Hungarian), Mat. Lapok. 11 (1960) 264–274.

[2] G.O.H. Katona, Combinatorial search problem, in: J.N. Srivastava et al., eds., A Survey of Combinatorial Theory (North-Holland, Amsterdam, 1973) 285–308.

[3] J. Spencer, Minimal completely separating systems, J. Combin. Theory 8 (1970) 446–447.